Topological classification of cellular automata

1991 J. Phys. A: Math. Gen. 24 L31

(http://iopscience.iop.org/0305-4470/24/1/007)

J. Phys. A: Math. Gen. **24** (1991) L31–L34. Printed in the UK

LETTER TO THE EDITOR

Topological classification of cellular automata

P-M Binder

Department of Theoretical Physics, University of Oxford, 1 Keble Road, Oxford OX1 3NP, UK

Received 14 August 1990, in final form 5 September 1990

Abstract. We summarize an extensive numerical study of basin and attractor sizes among the 88 distinct *elementary* cellular automata (CA) rules. Based on this study and on previous work in discretized dynamical systems, we propose a new classification of CA, complementary to that of Wolfram, in which attractor globality is important. With the use of fixed boundary conditions we find global periodic attractors in CA for the first time. Not a single instance of attractor chaos is observed in this class of rules.

In the past few years, cellular automata (CA) have become a popular subject of study for mathematicians, computer, physical, and natural scientists [1–5]. The purpose of this letter is to attempt to bridge the gap between CA and dynamical systems.

By comparing the behaviour of CA to that of discretized dynamical systems [6–8] we propose a new classification of dissipative CA, complementary to that proposed earlier by Wolfram [2, 9]. For example, consider the logistic equation $x_{n+1} = aAx_n(1-x_n)$, when x is discretized into 2^b states. Then, most of all 2^b initial conditions evolve to a single attractor, whose length may or may not depend on b. We call the property of *all* available initial conditions evolving to the same attractor for a given lattice or discretization size *globality*. Then, attractor *globality* is desirable, if CA are to behave similarly to continuous dynamical systems in the large-lattice limit. In order to find rules with global attractors, we do not need the concept of a distance or metric. In this sense, the new classification is topological.

In an extensive study of a family of 88 CA rules, to be defined below, we find mostly instances of multiple attractors, with rather small basins of attraction. We also see some examples of global fixed points, as well as periodic attractors when fixed boundary conditions are introduced. We, however, do not find a global attractor which grows with the system size, which would be the discrete equivalent of chaotic behaviour according to [6–8]. This leads us to conclude tentatively that previous observations of 'chaos' in elementary CA correspond to 'transient chaos' [10] or to evolution on a non-global attractor. Two important caveats are that we have studied a small sample of all the possible CA rules, and that large-lattice features may not have emerged yet for the lattice sizes of 14 or less used in the present study.

Cellular automata, originally proposed by Ulam and von Neumann [11], are systems discrete in space, time and state variable. All sites in a lattice have a microstate, belonging to a finite set, which is updated (usually synchronously) as a function of the microstates of a small neighbourhood around the site. One may interpret the sites as being placed contiguously in space, or alternatively, as being digits or bits which when properly weighed and added form the system's macrostate (a single number or label) at any time step. In the first case, the CA, being a system with many spatial degrees of freedom, is analogous to a partial differential equation or coupled map lattice. In the second case, being a system with one degree of freedom, it is analogous to an ordinary differential equation or iterated map.

The first approach, which corresponds loosely to a study of patterns, has been quite successful. For example, applications to hydrodynamics [12] have become a separate and fast-growing field. Wolfram [9] proposed a phenomenological classification of dissipative CA. His four classes correspond to (1) homogeneous states, (2) periodic or static non-interfering spatial patterns, (3) 'chaotic' behaviour, often accompanied by self-similar patterns, and (4) complex, often long-lived structures. It was noticed by Wolfram that the Hamming distance between nearby initial conditions grows in different ways for the four classes. This gives his classification a 'metric' footing. This classification has motivated much research on invariant distributions, long-range correlations, phase transitions, basic studies of complexity and mathematical analysis of CA rules [13].

Some advance has been made in the study of CA as dynamical systems as well. Standard methods are to study the structure of limit cycles and transients for different rules as a function of system size. The work has been mathematical [14] as well as numerical. In particular, [15] is an extensive numerical study of all distinct *elementary* CA rules. These are one-dimensional, with two states per site and three-site neighbourhoods, which gives a total of 256 rules. Of these, due to symmetries and 0–1 exchanges, only 88 are distinct. In this study, Peck has measured mean and maximum transient and cycle lengths, as well as the number of cycles and degree of dissipation as a function of lattice size.

It is instructive to compare Peck's results to the available studies of discrete dynamical systems [6–8]. Such studies have been performed for finite-state versions of well known dissipative maps such as the logistic and Hénon maps. The main results are as follows. (1) There is typically an almost global attractor, which attracts most initial conditions. This is also the case in the continuous map, and a common feature in dissipative dynamical systems, although problems with more than one basin of attraction (the long-time trajectories for such systems cannot be called attractors, as they are decomposable) have also been studied [16]. In this respect, as the discretization step becomes finer, the finite state approximation becomes better. (2) One can distinguish between regular and chaotic behaviour. The latter is not obvious, since a finite-state deterministic map iteration must end in a limit cycle of period no longer than the total number of available states. However, for periodic behaviour with small cycles, one finds that the attracting cycle is of constant size, independent of the number of states of the system. For chaotic behaviour, the cycle length grows as a power of the total number of states corresponding to the capacity dimension (typically one half). As pointed out in [6] even with fairly coarse discretization, one can distinguish already between regular and chaotic behaviour.

Peck's study is not satisfactory in two respects: First, although it enumerates cycle lengths, it gives no idea of the fraction of phase space attracted to each cycle (its basin size). Secondly, except for global fixed points, there seems to be a high multiplicity of basins, often related to the lattice size. These features are very discouraging in light of the results of [6–8].

We now summarize a recent study of the basin sizes of elementary CA. Since the cycle multiplicity in Peck's study seems to be related to the use of periodic boundary conditions, we have in addition investigated fixed boundary conditions: the four combinations of fixed zeros and ones to the left and right of the CA lattice. These boundary conditions (BCs) diminish the probability of cyclic permutations or shifts, identified by Jen [14] as a mechanism for multiple limit cycles; they can also 'generate' or 'absorb' information, as the rightmost bits and the binary point do in the iteration of the Bernoulli shift ($x_{n+1} = 2x_n$, mod 1).

The entire phase space for all 88 *elementary* rules and five BCs for lattice sizes of 9-12 was investigated. This involved iteration of *all* possible 2^9–2^{12} initial conditions for up to 2^9–2^{12} time steps, until a limit point or cycle is reached, and an exhaustive compilation of limit cycles and their basin sizes.

For the studied rules, lattice sizes and BCs (including practically all periodic BCs), we see in most cases a broken down phase space with several attractors; we often see that the *fraction* of phase space occupied by the largest basin decreases as the system size grows (for example, rules 2-7 and many others). We call this non-global behaviour class A; it is of little interest as a dynamical system. We must note that, while in some cases this happens in a regular fashion, in others the largest basin size oscillates, as in rule 60 for which it takes up 25, 12, 34 and 5 percent of the available phase space for $L = 9$–12 respectively. The present study can say very little about the infinite-lattice limit, and we suggest more detailed work on particular rules of interest. *Globally* attracting fixed points, which are of much more physical interest, will be called class B. All of Peck's observed rules can be put in one of these two classes. We find seventeen elementary rules which with at least one of the five possible boundary conditions belong to this class: rules 0, 2, 8, 14, 15, 24, 32, 34, 40, 42, 128, 130, 136, 138, 152, 160 and 162 in Wolfram's rule-numbering system [2].

We also see a dozen examples of global *periodic* attractors (of constant period 2, 3, 4 or 6 for lattice sizes up to 14), once the fixed BCs have been used. We call this behaviour type C; it corresponds to periodic behaviour in discretized dynamical systems [6–8]. It has *never* been observed before in CA. The particular rules which, for at least one boundary condition show this class C behaviour are rules 2, 3, 10, 11, 24, 27, 34, 35, 42, 56, 57, 58, 130 and 152 in Wolfram's numbering system.

Surprisingly, no global attractors of length growing with the lattice size were found in this study. Since this is the description of attractor chaos in discretized dynamical systems, the conclusion is that the 'chaos' reported in the CA literature may correspond to long transients, leading to a fixed point or limit cycle, or to evolution along a non-global limit cycle which cannot be called an 'attractor' in the dynamical systems sense. This finding is supported by concurrent work of Langton [13] and of Gallas and Herrmann [17], who have identified Wolfram's class 3 and 4 behaviour respectively as transient phenomena.

In this letter we have proposed a classification of dissipative CA which takes into account how a rule acts on all possible initial conditions, instead of just looking at the shape of patterns resulting from particular rules and initial conditions. In particular, we put rules in which *all* initial conditions evolve to a single attractor in separate classes; this classification is consistent with what is known about finite-state iterations of dissipative maps. It is topological in the sense that the notion of distance is never used, and therefore complementary to Wolfram's classification. In the new classification, most rules either lead to a broken-down phase space (class A), or to a global fixed point (class B). By using non-periodic boundary conditions we obtain for the first time global CA attractors of period other than one, independently of lattice size (class C). Another possible class, D, corresponding to a global attractor of length growing with the system size, was not found in elementary CA. This class would correspond to true attractor chaos. Classes B, C and D are found in discretized versions of dissipative iterations. This classification is tentative: rules with two (or more) large basins which occupy a constant, significant fraction of phase space for all CA lattice sizes, or rules with almost-global attractors could be considered as belonging to a class other than A. Also, since we only considered lattice sizes of 14 or less, a number of large-lattice effects could appear. Detailed studies of particular rules, such as those that have been done for rule 22 by Grassberger [13] and Zabolitzky [18] can provide more information about this.

Two interesting questions remain, which we hope will be addressed by other investigators. Are there non-elementary CA rules which belong to class D? Can a proof be provided for the existence of global periodic attractors for all lattice sizes?

We thank H Gutowitz, E Jen and C G Langton for useful discussions, an anonymous referee for several suggestions on the manuscript, and the SERC for financial support.

References

[1] Farmer J D, Toffoli T and Wolfram S (eds) 1984 *Cellular Automata* (Amsterdam: North-Holland)

[2] Wolfram S (ed) 1986 *Cellular Automata: Theory and Applications* (Singapore: World Scientific)

[3] Toffoli T and Margolus N H 1987 *Cellular Automata Machines* (Cambridge, MA: MIT Press)

[4] Manneville P *et al* (eds) 1989 *Cellular Automata and the Modeling of Complex Systems* (Berlin: Springer)

[5] Jen E and Gutowitz H A (eds) 1990 *Proc. 1989 NATO Workshop on Cellular Automata* (Amsterdam: North-Holland)

[6] Binder P M and Jensen R V 1986 *Phys. Rev.* A **34** 4460–3

[7] Beck C and Roepstorff G 1987 *Physica* **25D** 173

[8] Grebogi C, Ott E and Yorke J A 1988 *Phys. Rev.* A **38** 3688

[9] Wolfram S 1983 *Rev. Mod. Phys.* **55** 601

[10] Crutchfield J P and Kaneko K 1988 *Phys. Rev. Lett.* **60** 2715
Grebogi C, Ott E and Yorke J A 1983 *Physica* **7D** 181

[11] von Neumann J 1966 *Theory of Self-reproducing Automata* ed A Burks (Urbana, IL: University of Illinois Press)

[12] Complex systems vol 1, no 4 (August 1987)
Monaco R (ed) 1989 *Discrete Kinetic Theory, Lattice Gas Dynamics and Foundations of Hydrodynamics* (Singapore: World Scientific)

[13] Jen E 1986 *J. Stat. Phys.* **43** 219
Grassberger P 1986 *J. Stat. Phys.* **45** 27
Gutowitz H A, Victor J D and Knight B W 1987 *Physica* **28D** 18
Chaté H and Manneville P 1987 *Phys. Rev. Lett.* **58** 112
Langton C G 1989 *PhD Thesis* University of Michigan

[14] Martin O, Odlyzko A and Wolfram S 1984 *Commun. Math. Phys.* **93** 219
Jen E 1988 *Commun. Math. Phys.* **119** 13

[15] Peck H 1986 *Cellular Automata: Theory and Applications* pp 531–45

[16] Grebogi C, Ott E and Yorke J A 1987 *Physica* **24D** 243

[17] Gallas J A G and Herrmann H J 1990 *Int. J. Mod. Phys.* C **1** 181

[18] Zabolizky 1988 *J. Stat. Phys.* **50** 1255